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NEW INTEGRABLE CASES OF EULER'S EQUATIONS*

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It is proved that the equations of rotation of a solid fixed at its centre of mass in a Newtonian field of an arbitrary fairly distant object are completely Liouville integrable. The integrable case of rotation of a magnetized solid in a uniform gravitational and magnetic field, which generalizes the Kovalevskaya case, is indicated.

1. The Liouville integrability of the rotation of an arbitrary solid fixed at its centre of mass in a Newtonian field of an arbitrarily distant object.

The investigation of the rotation of an arbitrary solid T fixed at the centre O of its mass in a Newtonian field of an arbitrary fairly distant object V reduces to the investigation of the rotation of the solid T in a Newtonian field with an arbitrary uniform quadratic potential /1, 2/. Until recently only a single case of this problem was known. In it the gravitational field of the object V was axially symmetric (the axes passes through the point O) and the quadratic potential φ is equivalent to the potential $\varphi = a(x^1)^2$ Brun's problem /3/, see also /1, 2, 4/). The equations which describe the rotation of a solid T reduce to the integrable Klebsch problem for Kirchhoff's equations. It is shown below that this problem is, in the most general case, completely Liouville integrable.

The equations of rotation of a solid T about a fixed point O are considered in a reference system S rigidly attached to the solid. Let $\varphi(x^1, x^2, x^3)$ be the Newtonian potential in the stationary reference system F whose centre is at the point O, and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the three unit vectors of the stationary system of coordinates in the reference system S. Let us determine the potential function

$$U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \int_T \rho(\mathbf{r})\, \varphi((\mathbf{r}, \boldsymbol{\alpha}), (\mathbf{r}, \boldsymbol{\beta}), (\mathbf{r}, \boldsymbol{\gamma}))\, dr^1\, dr^2\, dr^3 \tag{1.1}$$

where $\rho(\mathbf{r})$ is the density of the solid T at the point $\mathbf{r}$. The equations of rotation of the solid T in the Newtonian field with potential $\varphi(x^1, x^2, x^3)$ in the reference system S have the form

$$\begin{aligned} &\mathbf{M}^{\cdot} = \mathbf{M} \times \boldsymbol{\omega} + (\partial U/\partial \boldsymbol{\alpha}) \times \boldsymbol{\alpha} + (\partial U/\partial \boldsymbol{\beta}) \times \boldsymbol{\beta} + (\partial U/\partial \boldsymbol{\gamma}) \times \boldsymbol{\gamma}, \\ &\boldsymbol{\alpha}^{\cdot} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \boldsymbol{\beta}^{\cdot} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \boldsymbol{\gamma}^{\cdot} = \boldsymbol{\gamma} \times \boldsymbol{\omega} \end{aligned} \tag{1.2}$$

where $\mathbf{M}, \boldsymbol{\omega}$ are the vectors of kinetic momentum and angular velocity, whose coordinates are connected by the relations

$$M_i = \sum_{k=1}^{3} I_{ik}\omega_k, \qquad I_{ik} = \int_T \rho(\mathbf{r}) \left(\delta_{ik} \sum_{l=1}^{3} (r^l)^2 - r^i r^k\right) dr^1\, dr^2\, dr^3 \tag{1.3}$$

where I_{ik} are the components of the inertia tensor of the solid T in system S.

Theorem 1. The equations of rotation of an arbitrary solid around a fixed point O in a Newtonian field with an arbitrary quadratic potential

$$\varphi(x_1, x_2, x_3) = \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} x^i x^j \tag{1.4}$$

are completely Liouville integrable.

Note that Newtonian fields with a potential of the form (1.4) that satisfy the Laplace equation $\Delta\varphi(x^1, x^2, x^3) = 0$, are subject to the condition $a_{11} + a_{22} + a_{33} = 0$.

We select the unit vectors of the fixed reference system F that coincide with the principal axes of the quadratic form and, after such a transformation, obtain $2\varphi(x) = a_1(x^1)^2 + a_2(x^2)^2 + a_3(x^3)^2$. The unit vectors of the reference system S are selected so that they coincide with the principal axes of the inertia tensor, i.e. $I_{ik} = I_i\delta_{ik}$. The potential function U (1.1) then takes the form

$$\begin{aligned} 2U = U_0 - a_1(I_1\alpha_1^2 + I_2\alpha_2^2 + I_3\alpha_3^2) - a_2(I_1\beta_1^2 + I_2\beta_2^2 + \\ I_3\beta_3^2) - a_3(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2), \quad U_0 = (a_1 + a_2 + a_3)(I_1 + I_2 + I_3)/2 \end{aligned} \tag{1.5}$$

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Taking advantage of the isomorphism of the vectors with components v^i in R^3 and the skew symmetry (3×3) of matrices with elements V_{jk}, we have

$$v^i \to V_{jk} = -\sum_{i=1}^{3} v^i \varepsilon_{ijk} \tag{1.6}$$

for which the vector product $\mathbf{x} \times \mathbf{y}$ becomes the commutator of the matrices $[X, Y] = XY - YX$. After this isomorphism the matrices $\alpha, \beta, \gamma, M, \omega$ correspond to the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\mathbf{M}$, $\boldsymbol{\omega}$, and Eqs.(1.2) take the form

$$\begin{aligned} M^{\cdot} &= [M,\ \omega] + a_1[\alpha,\ C\alpha + \alpha C] + a_2[\beta,\ C\beta + \beta C] + \\ &\quad a_3[\gamma,\ C\gamma + \gamma C] \\ \alpha^{\cdot} &= [\alpha,\ \omega],\quad \beta^{\cdot} = [\beta,\ \omega],\quad \gamma^{\cdot} = [\gamma,\ \omega] \end{aligned} \tag{1.7}$$

where the matrix C has elements $C_{ij} = (2^{-1}(I_1 + I_2 + I_3) - I_i)\,\delta_{ij}$ and $M_{ij} = I_k\omega_{ij}\,(i, j, k = 1, 2, 3)$. By virtue of (1.7) we have

$$(\alpha^2)^{\cdot} = [\alpha^2, \omega],\quad (\beta^2)^{\cdot} = [\beta^2, \omega],\quad (\gamma^2)^{\cdot} = [\gamma^2, \omega]$$

We introduce the matrix $u = a_1\alpha^2 + a_2\beta^2 + a_3\gamma^2$ and use the obvious identity $[x, Cx + xC] = [x^2, C]$; from (1.7) we have the corollary

$$M^{\cdot} = [M,\ \omega] + [u,\ C],\quad u^{\cdot} = [u,\ \omega] \tag{1.8}$$

The matrices M and ω are skew symmetric, while the matrices u and C are symmetric. Another derivation of (1.8) in a more general case is indicated in Sect.2.

Equations (1.8) are Euler equations in the adjoint space of the Lie algebra L_9', whose elements l have the form $l = M + u$, where M, u are three-dimensional matrices, and the commutators $M^t = -M$, $u^t = u$ are defined by the conditions

$$[M,\ u] = Mu - uM,\ [M_1,\ M_2] = M_1M_2 - M_2M_1,\ [u_1,\ u_2] = 0 \tag{1.9}$$

The action orbits X of the respective Lie group G_9' and $L_9'^*$ are simplectic manifolds, and $V^6 = R^3 \times \mathrm{SO}(3) = T\,(\mathrm{SO}(3))$ is a bunch tangent ot the Lie group $\mathrm{SO}(3)$. The manifolds V^6 are defined by the conditions $\lambda_j(u) = \mathrm{const}$, where $\lambda_j(u)$ are the eigennumbers of the matrix u, if $\lambda_1 = \lambda_2 \neq \lambda_3$, then orbit $X = V^5 = R^3 \times S^2$, and $X = R^3$, if $\lambda_1 = \lambda_2 = \lambda_3$. The Poisson brackets of the functions on $L_9'^*$ are determined by the formulae

$$\{f, g\} = \sum_{i,j,k} C_{ij}{}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \tag{1.10}$$

where $C_{ij}{}^k$ are structural constants of the Lie algebra L_9' in basis x_i. The Poisson brackets (1.10) on the submanifolds of V^6 are non-degenerate.

Equations (1.8) are of Hamiltonian form

$$M_{ij}{}^{\cdot} = \{M_{ij},\ H\},\quad u_{ij}{}^{\cdot} = \{u_{ij},\ H\} \tag{1.11}$$

where the Hamiltonian is $H = J_1 = \mathrm{Tr}\,(2^{-1}M\cdot\omega + u\cdot C)$.

We will introduce a matrix B with elements $B_{ij} = I_1I_2I_3I_i^{-1}\delta_{ij}$. System (1.8) has two supplementary first integrals

$$J_2 = \mathrm{Tr}\,(2^{-1}M^2 + Bu),\quad J_3 = \mathrm{Tr}\,(M^2u + Bu^2) \tag{1.12}$$

The integrals J_1, J_2, J_3, are obviously functionally independent. By virtue of (1.11) we have $J_2{}^{\cdot} = \{J_2, J_1\} = 0$, $J_3{}^{\cdot} = \{J_3, J_1\} = 0$. Direct calculation shows that the Poisson bracket $\{J_2, J_3\} = 0$, i.e. the three integrals J_1, J_2, J_3 are in involution. Hence the Hamiltonian system (1.8)–(1.11) is completely Liouville integrable on the six-dimensional simplectic submanifolds of V^6. The trajectories of system (1.8)–(1.11) are quasiperiodic windings of three-dimensional tori T^3 in the space $L_9'^*$ defined by the conditions $J_i = c_i$, $\lambda_j(u) = k_j$.

Equations (1.8) have an equivalent representation in the form of the Lax matrix equation, which depends on an arbitrary spectral parameter E

$$L^{\cdot} = [L, Q],\quad L = BE^2 + ME + u,\quad Q = \omega - EI \tag{1.13}$$

Integrals (1.12) are the coefficients at E^2 in the expansion of the functions $\mathrm{Tr}\,(L^2(E))$ and $\mathrm{Tr}\,(L^3(E))$ (which by virtue of (1.13) are independent of t) in powers of the parameter E. Owing to the existence of representation (1.13) the Euler equations are explicitly integrable with respect to theta-functions of Riemannian surfaces specified by the equations $R\,(E,\,w) = \det\,(L\,(E) - w\cdot 1) = 0$. The Lax equations with a spectral parameter were studied (in connection with other problems) in /5, 6/.

Note that in the integrable Klebsch case /1, 2, 4/ which describes the solution of Brun's problem /3/ $(\varphi = a\,(x^1)^2)$, the dynamics of trajectories is quasiperiodic on two-dimensional tori T^2. In little known publications by Brun /7, 8/ two supplementary integrals of the equations of a solid rotating in a filed with a general quadratic potential of the form (1.4) are indicated. Goryachev /9/ found two supplementary integrals in the case of potential $\varphi(x) = a((x^1)^2 - (x^2)^2)$. In these works the Hamiltonian structure of Euler's equations (1.2), and the question of their Liouville integrability was not considered. In the problem investigated here the combined level of all six first integrals $J_i = c_i$, $\lambda_j(u) = k_j$ is a three-dimensional manifold, as in the most general case of Euler-Poisson equations. Thus, unlike the integrable Euler-Poisson equations, the method of the last Jacobi multiplier is not applicable.

The first proof of Theorem 1, different from the one given here, appeared in /10/. In the present paper in addition to the Liouville proof of integrability based on investigation of the Hamiltonian structure of (1.8), a proof is obtained of the integrability of these equations in terms of Riemann theta-functions, which is a corollary of the representation of system (1.8) in the forms (1.13). The explicit formulae expressing the angular velocities of the solid $\omega^i(t)$ in terms of Riemann theta-functions were derived in /11/.

The general equations (1.2) which describe the rotation of a solid around a fixed point in a Newtonian field are Euler equations in the conjugate space L_{12}^* to the Lie algebra L_{12}, whose commutators in the basis X_i, Y_j^α $(i, j, k, \alpha, \beta = 1, 2, 3)$ have the form

$$[X_i, X_j] = \varepsilon_{ijk}X_k, \quad [X_i, Y_j^\alpha] = \varepsilon_{ijk}Y_k^\alpha, \quad [Y_i^\alpha, Y_j^\alpha] = 0 \tag{1.14}$$

Equations (1.2) have an energy integral $J_1 = 2^{-1}(\mathbf{M}, \boldsymbol{\omega}) - U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and six geometric integrals $J_2, \ldots, J_7$, which determine the paired constant scalar products of the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. The combined level of the integrals $J_2, \ldots, J_7$ is a submanifold of $V_1^6 = T\,(\mathrm{SO}(3))$. Equations (1.2) on V_1^6 are of Hamiltonian form with the Hamiltonian J_1 which is of simplectic structure determined by (1.10)–(1.14). Equations (1.2) are the first example of the physically important Euler equations of the Lie algebras, where the non-linearity of the potential function U in the Hamiltonian J_1 may be as complicated as desired.

2. Integrable cases of a solid rotating round a fixed point in a force field with a quadratic potential.

Consider the rotation of a solid T round a fixed point in a force field with the potential

$$\varphi = \frac{1}{2}\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}(|x|)\,x^\alpha x^\beta, \qquad |x| = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2} \tag{2.1}$$

where $a_{\alpha\beta}(|x|)$ are arbitrary differential functions of the variable $|x|$. The Newtonian potentials of the form (2.1) that satisfy the Laplace equation $\Delta\varphi = 0$, are determined by the formulae

$$\varphi = \Big(\sum_{\alpha,\beta=1}^{3} c_{\alpha\beta}x^\alpha x^\beta\Big)|x|^{-5} + \sum_{\alpha,\beta=1}^{3} b_{\alpha\beta}x^\alpha x^\beta + c\,|\mathbf{x}|^{-1}$$

$$c_{11} + c_{22} + c_{33} = 0, \quad b_{11} + b_{22} + b_{33} = 0$$

We introduce the following four-component tensor symmetric about two pairs of indices α, β and i, k which generalizes the inertia tensor I_{ik} (1.3)

$$T_{\alpha\beta ik} = \int_T \rho(\mathbf{r})\,a_{\alpha\beta}(|\mathbf{r}|)\Big(\delta_{ik}\sum_{l=1}^{3}(r^l)^2 - r^i r^k\Big)\,dr^1\,dr^2\,dr^3 \tag{2.2}$$

Theorem 2. If the tensor $T_{\alpha\beta ik}$ can be represented in the form

$$T_{\alpha\beta ik} = A_{\alpha\beta}I_{ik} + B_{\alpha\beta}\delta_{ik} + \delta_{\alpha\beta}C_{ik} \tag{2.3}$$

where A, B, C are arbitrary symmetric matrices, the equations of rotation of the solid T around a fixed point O $(x^i = 0)$ in a field with potential (2.1) is completely Liouville integrable.

If $a_{\alpha\beta}(|x|) = \mathrm{const}$, then conditions (2.3) are obviously satisfied, and hence Theorem 2 generalizes Theorem 1. If the solid T is a sphere whose density is

$$\rho(\mathbf{r}) = \rho_1(\mathbf{r}/|\mathbf{r}|)\rho_2(|\mathbf{r}|) \tag{2.4}$$

then conditions (2.3) are satisfied for arbitrary functions $a_{\alpha\beta}(|\mathbf{x}|)$ (then $B_{\alpha\beta} = C_{ik} = 0$), hence the equations of rotation of such a solid in a field with an arbitrary potential of form (2.1) are completely Liouville integrable.

Let the orthogonal matrix $Q(t)$ define the transformation from the Lagrangian coordinates r^k attached to the frame of reference S in Euler coordinates x^i: $x^i = Q_k^{\ i}(t)r^k$ (recurrent indices indicate summation everywhere). By definition we have $Q^\cdot = Q\omega$, where ω is the angular velocity matrix.

In the system S the potential (2.1) has the form

$$\varphi = \frac{1}{2}\sum_{\alpha,\beta,n,l} a_{\alpha\beta}(|\mathbf{r}|)\,Q_n^{\ \alpha}Q_l^{\ \beta}r^n r^l$$

The components of the moment of forces acting on the solid T in a field with potential φ (2.2) are given by

$$K_i = \int_T \left(\mathbf{r} \times \rho \frac{\partial\varphi}{\partial \mathbf{r}}\right)_i dr^1\, dr^2\, dr^3 = \int_T \varepsilon_{imn} r^m \rho a_{\alpha\beta}(|\mathbf{r}|) Q_n{}^\alpha Q_l{}^\beta r^l\, dr^1\, dr^2\, dr^3 \tag{2.5}$$

The skew symmetric matrix K which by the isomorphism (1.6) corresponds to the vector of the moment of forces (2.5) has the elements

$$K_{jk} = -\varepsilon_{ijk} K_i = T_{\alpha\beta jl} Q_k{}^\alpha Q_l{}^\beta - T_{\alpha\beta kl} Q_j{}^\alpha Q_l{}^\beta \tag{2.6}$$

Substituting (2.3) into formulae (2.6), we obtain

$$K_{jk} = I_{jl} A_{\alpha\beta} Q_k{}^\alpha Q_l{}^\beta - I_{kl} A_{\alpha\beta} Q_j{}^\alpha Q_l{}^\beta \tag{2.7}$$

We introduce the matrix $u = Q^t A Q$. In matrix form (2.7) means that $K = Iu - uI = -[u, I]$. Hence the equations that define the angular momentum matrix and (by virtue of $Q^\cdot = Q\omega$) of matrix u have the form

$$M^\cdot = [M, \omega] - [u, I], \quad u^\cdot = [u, \omega] \tag{2.8}$$

Equations (2.8) completley define the rotation of a solid in a field with potential (2.1), if conditions (2.3) are satisfied. They obviously are the same as (1.8), and are therefore completely Liouville integrable.

3. The integrable case of the rotation of a magnetized solid in a uniform gravitational and magnetic field. Consider the rotation of the solid T round a fixed point O, whose magnetic moment $\mathbf{m}$ is constant in a uniform gravitational and magnetic field. Let us assume that the inertia tensor of the solid in the rotating frame of reference S is diagonal with components I_1, I_2, I_3. The directions of the uniform gravitational and magnetic field stress vectors are defined by the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ of unit length; the vector $\mathbf{r}$ defines the position of the centre of mass (in system S), m is the mass of the solid, and g and h are the intensities of the gravitational and magnetic fields; $\mathbf{M}$ and $\boldsymbol{\omega}$ are the angular momentum and angular velocity vector $M_k = I_k\omega_k$. The equations of motion in system S have the form

$$\mathbf{M}^\cdot = \mathbf{M} \times \boldsymbol{\omega} + mg\mathbf{r} \times \boldsymbol{\gamma} + h\mathbf{m} \times \boldsymbol{\delta}, \quad \boldsymbol{\gamma}^\cdot = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad \boldsymbol{\delta}^\cdot = \boldsymbol{\delta} \times \boldsymbol{\omega} \tag{3.1}$$

The equations of rotation about a fixed point of a fully charged solid with total charge σ in a constant gravitational and electric field are of the same form, except that instead of h we have E (the electric field strength), and instead of $\mathbf{m}$ we have the dipole moment vector

$$\mathbf{d} = \int_T \sigma(\mathbf{r})\, \mathbf{r}\, dr^1\, dr^2\, dr^3$$

where $\sigma(\mathbf{r})$ is the electric charge density.

Equations (3.1) have the following first integrals:

$$J_1 = 2^{-1}(\mathbf{M}, \boldsymbol{\omega}) - mg(\mathbf{r}, \boldsymbol{\gamma}) - h(\mathbf{m}, \boldsymbol{\delta}), \quad J_2 = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) \tag{3.2}$$
$$J_3 = (\boldsymbol{\delta}, \boldsymbol{\delta}), \quad J_4 = (\boldsymbol{\gamma}, \boldsymbol{\delta})$$

The integral J_1 is identical with the total energy of the solid. The manifold V^6, defined by conditions $J_2 = c_2$, $J_3 = c_3$, $J_4 = c_4$, is generally homeomorphic to the product $V^6 = R^3 \times SO(3) = T\,(SO\,(3))$.

Equations (3.2) are the Euler equations in the conjugate space $L_9{}^*$ of the Lie algebra L_9, whose commutators in the basis X_i, $Y_j{}^\alpha$ have the form (1.14), where $\alpha, \beta = 1, 2$. The three-dimensional vectors $\mathbf{M}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ belong to the subspaces $X_i{}^*, Y_i{}^{1*}, Y_i{}^{2*}$ respectively.

The Poisson brackets of the functions in space L_9^* are defined by the formulae (1.10). For the basis functions M_i, γ_j, δ_k we obtain

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{M_i, \delta_j\} = \varepsilon_{ijk}\delta_k \tag{3.3}$$
$$\{\gamma_i, \gamma_j\} = \{\delta_i, \delta_j\} = \{\gamma_i, \delta_j\} = 0$$

By virtue of (1.10) the Poisson brackets of arbitrary polynomials of M_i, γ_j, δ_k are calculated by the Leibniz rule using (3.3). The functions J_2, J_3, J_4 cancel the Poisson brackets in (3.3); the manifolds of their levels V^6 have a non-degenerate simplectic structure (these structures are similar to those in /12/ for the Kirchhoff equations). Equations (3.1) have the Hamiltonian form

$$M_i^\cdot = \{M_i, H\}, \quad \gamma_j^\cdot = \{\gamma_j, H\}, \quad \delta_k^\cdot = \{\delta_k, H\} \tag{3.4}$$

where the Hamiltonian $H = J_1$.

Theorem 3. Equaties (3.1) with the conditions

$$mg\mathbf{r} = (R, 0, 0), \quad h\mathbf{m} = (0, Q, 0), \quad I_1 = I_2 = 2I_3 \tag{3.5}$$

have the first integral

$$J_5 = z_1^2 + z_2^2 \tag{3.6}$$
$$z_1 = M_1^2 - M_2^2 + 4I_3R\gamma_1 - 4I_3Q\delta_2, \quad z_2 = 2M_1M_2 + 4I_3R\gamma_2 + 4I_3Q\delta_1$$

On the manifold $J_5 = 0$ $(z_1 = 0,\ z_2 = 0)$ equations (3.1) have the supplementary integral $J_6 = \{z_1, z_2\}$, and are completely Liouville integrable.

A direct check shows that (3.1) yields the equations

$$z_1^{\cdot} = \{z_1, H\} = I_3^{-1}M_3z_2, \quad z_2^{\cdot} = \{z_2, H\} = -I_3^{-1}M_3z_1 \tag{3.7}$$

that prove the existence of integral J_5. Equations (3.7) are equivalent to the single equation $z^{\cdot} = -iI_3^{-1}M_3z$, where $z = z_1 + iz_2$, and $J_5 = |z|^2$.

The manifold of level $J_5 = 0$ $(z_1 = 0,\ z_2 = 0)$ in intersection with the submanifolds of V^6 determines the simplectic four-dimensional submanifolds V^4 (the induced simplectic structure is non-degenerate). System (3.1)−(3.4) has on submanifolds V^4 the supplementary first integral

$$J_6 = \{z_1, z_2\} = 4M_3(M_1^2 + M_2^2 + 4I_3RM_1\gamma_3 + 4I_3QM_2\delta_3) \tag{3.8}$$

In fact, by virtue of the Jacobi identity and (3.7), we have

$$J_6^{\cdot} = \{\{z_1, z_2\}, H\} = -\{\{z_2, H\}, z_1\} + \{\{z_1, H\}, z_2\} =$$
$$I_3^{-1}z_1\{M_3, z_1\} + I_3^{-1}z_2\{M_3, z_2\} = I_3^{-1}(2z_1M_1M_2 - z_2(M_1^2 - M_2^2))$$

Consequently on submanifolds V^4 $(z_1 = z_2 = 0,\ J_2 = c_2,\ J_3 = c_3,\ J_4 = c_4)$ we have $J_6^{\cdot} = 0$. Thus the Hamiltonian system (3.1)−(3.4) has on invariant submanifolds V^4 a supplementary manifold J_6 and is completely Liouville integrable. When there is no magnetic field $(Q = 0)$, the integrable case of (3.1) becomes the classic Kovalevskaya case.

In the integrable case obtained the potential function $U = R\gamma_1 + Q\delta_2$ (see (1.2)) essentially depends on three Euler angles φ, ψ, θ. Integrable cases in which the function U depends only on two Euler angles φ, θ, were investigated in /13, 14/.

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